Orthocentre of a Triangle

Introduction:

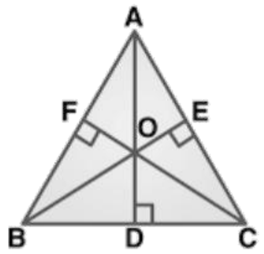
In geometry, the orthocentre is an important concept that arises when studying triangles. It is one of the key points associated with any triangle, along with the centroid, circumcentre, and incentre. The orthocentre holds particular significance in various branches of geometry, especially when dealing with the properties of triangle altitudes. The orthocentre is the point where all three altitudes of a triangle meet, and understanding its properties and applications can enhance one's knowledge of both Euclidean geometry and more advanced topics such as triangle centres.

What is the Orthocentre?

The **orthocentre** of a triangle is the point where the three **altitudes** of the triangle intersect. In simple terms, an altitude of a triangle is a perpendicular segment drawn from a vertex to the opposite side (or the line containing the opposite side). Each triangle has three altitudes, and their common point of intersection is called the orthocentre.

Mathematically, the orthocentre is defined as:

- The orthocentre is the point of intersection of the altitudes of the triangle.
- The altitudes can be inside or outside the triangle, depending on whether the triangle is acute, right, or obtuse.



Types of Triangles and the Position of the Orthocentre:

The position of the orthocentre varies based on the type of triangle:

1. **Acute Triangle**: In an acute triangle (where all angles are less than 90°), the orthocentre lies **inside** the triangle.
2. **Right Triangle**: In a right triangle (where one angle is exactly 90°), the orthocentre lies **on** the right-angle vertex of the triangle. This is because the altitudes from the two acute angles are perpendicular to the legs of the right triangle, and their intersection is the vertex of the right angle.
3. **Obtuse Triangle**: In an obtuse triangle (where one angle is greater than 90°), the orthocentre lies **outside** the triangle. This occurs because the altitudes from the acute angles are extended outside the triangle to meet.

Altitudes and How to Construct Them:

To understand the orthocentre more clearly, it is essential to understand how altitudes are constructed in a triangle:

- Altitude from Vertex A: Draw a line from vertex A that is perpendicular to the opposite side BC. This line is called the altitude from vertex A.
- Altitude from Vertex B: Similarly, draw a line from vertex B that is perpendicular to side AC. This is the altitude from vertex B.
- Altitude from Vertex C : Finally, draw a line from vertex C that is perpendicular to side AB. This is the altitude from vertex C.

The point where these three altitudes meet is the orthocentre.

Example of Finding the Orthocentre:
Let's consider a simple example of finding the orthocentre for a triangle with known coordinates.
Example: Find the orthocentre of a triangle with vertices $A(1,2), B(4,6)$, and $C(7,1)$.
Solution:
Step 1: Find the equation of the altitudes.
To find the equation of the altitude from vertex A, we first need the slope of line BC. The slope of line BC is:

$$\text{Slope of BC} = \frac{y_C - y_B}{x_C - x_B} = \frac{1-6}{7-4} = \frac{-5}{3}$$

The slope of the altitude from A will be the negative reciprocal of the slope of line BC, so:

$$\text{Slope of altitude from A} = \frac{3}{5}$$

Using the point-slope form of the equation of a line, the equation of the altitude from vertex $A(1,2)$ is:

$$y - 2 = \frac{3}{5}(x-1)$$

Simplifying:

$$y = \frac{3}{5}x + \frac{7}{5}$$

Step 2: Repeat for other altitudes.
Repeat the same process for the altitude from vertex B and C to get their respective equations.
Step 3: Solve for the intersection.
Once you have the equations of the altitudes, solve the system of equations to find the coordinates of the orthocentre. This point will be the intersection of the three altitudes.

Properties of the Orthocentre:

1. **Collinearity with Other Triangle Centres**: The orthocentre, centroid, and circumcentre of a triangle are always collinear. This line is called the **Euler line**. The centroid divides the Euler line in a 2:1 ratio, with the centroid being closer to the orthocentre.

2. **Symmetry**: The orthocentre exhibits symmetry with the circumcentre of the triangle. In fact, if we reflect the orthocentre over any side of the triangle, the reflected point will lie on the circumcircle of the triangle.
3. **Relation with Triangle's Circumcentre**: The orthocentre can be used to find properties of the circumcentre, which is the centre of the circle that passes through all three vertices of the triangle (circumcircle). The orthocentre provides insight into the configuration of the triangle's circumcircle.
4. **Inversion**: Under an inversion with respect to the circumcircle of the triangle, the orthocentre and the circumcentre swap places.
5. **Altitude Lengths:** The altitudes from the vertices of the triangle are important for calculating the area of the triangle. The area A of a triangle can be expressed using any of the altitudes as:
   $A = \frac{1}{2} \times \text{Base } \times \text{Height}$
6. **Triangle Type Influence**: The position of the orthocentre helps determine the type of triangle (acute, obtuse, or right), as mentioned earlier.

Applications of the Orthocentre

1. **Geometric Constructions**: The orthocentre is often used in geometric constructions, where finding the point of intersection of perpendiculars helps solve complex problems related to triangle properties.
2. **Triangle Centres and Geometry**: The orthocentre is one of the key points used to study the relationship between various centres of a triangle. The centroid, incentre, circumcentre, and orthocentre provide a framework for understanding the properties of triangles and solving geometric problems.
3. **Area and Perimeter Calculations**: The altitudes of a triangle, which converge at the orthocentre, are essential in calculating the area of the triangle, which is fundamental in many practical applications, such as engineering and architecture.
4. **Trigonometry**: The orthocentre plays a role in trigonometric applications involving the angles of a triangle. The perpendiculars (altitudes) help in trigonometric relations and can be used to compute various geometric properties like the angles and side lengths of a triangle.
5. **Optimization Problems**: In advanced geometry, the orthocentre can be used in optimization problems where we need to find specific geometric configurations or optimize certain properties of a triangle, such as minimizing the area or maximizing symmetry.

Frequently Asked Questions (FAQs):

1. What happens to the orthocentre in an equilateral triangle?

Ans: In an equilateral triangle, the orthocentre coincides with the centroid, circumcentre, and incentre. All these points lie at the same location, which is the centre of the triangle.

2. How is the orthocentre different from the centroid?
Ans: The centroid is the point where the three medians of a triangle meet, and it is always inside the triangle. The orthocentre, on the other hand, is the point where the altitudes meet. The centroid divides the Euler line in a 2:1 ratio, with the centroid being closer to the orthocentre.

3. Can the orthocentre be outside the triangle?
Ans: Yes, the orthocentre can be outside the triangle, specifically when the triangle is obtuse. In an obtuse triangle, the altitudes from the two acute angles extend outside the triangle, and their intersection point (the orthocentre) lies outside the triangle.

4. Can the orthocentre be used to find the area of a triangle?
Ans: Yes, the orthocentre is closely related to the altitudes of the triangle, and the altitudes are essential in calculating the area of the triangle. The area A of the triangle can be calculated using the formula:

$$A = \frac{1}{2} \times \text{Base } \times \text{Height}$$

where the base is any side of the triangle, and the height is the perpendicular distance from the opposite vertex to the base.